Degree powers in graphs with forbidden subgraphs

Béla Bollobás[*†‡] and Vladimir Nikiforov[*]

February 1, 2008

Abstract

For every real $p > 0$ and simple graph G, set

$$f(p, G) = \sum_{u \in V(G)} d^p(u),$$

and let $\phi(p, n, r)$ be the maximum of $f(p, G)$ taken over all K_{r+1}-free graphs G of order n. We prove that, if $0 < p < r$, then

$$\phi(p, n, r) = f(p, T_r(n)),$$

where $T_r(n)$ is the r-partite Turan graph of order n. For every $p \geq r + \lceil \sqrt{2r} \rceil$ and n large, we show that

$$\phi(p, n, r) > (1 + \varepsilon) f(p, T_r(n))$$

for some $\varepsilon = \varepsilon(r) > 0$.

Our results settle two conjectures of Caro and Yuster.

1 Introduction

Our notation and terminology are standard (see, e.g. [1]).

Caro and Yuster [3] introduced and investigated the function

$$f(p, G) = \sum_{u \in V(G)} d^p(u),$$

where $p \geq 1$ is integer and G is a graph. Writing $\phi(r, p, n)$ for the maximum value of $f(p, G)$ taken over all K_{r+1}-free graphs G of order n, Caro and Yuster stated that, for every $p \geq 1$,

$$\phi(r, p, n) = f(p, T_r(n)), \tag{1}$$

[*]Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA
[†]Trinity College, Cambridge CB2 1TQ, UK
[‡]Research supported in part by DARPA grant F33615-01-C-1900.

where $T_r(n)$ is the r-partite Turán graph of order n. However, simple examples show that (1) fails for every fixed $r \geq 2$ and all sufficiently large p and n; this was observed by Schelp [4]. A natural problem arises: given $r \geq 2$, determine those real values $p > 0$, for which equality (1) holds. Furthermore, determine the asymptotic value of $\phi(r, p, n)$ for large n.

In this note we essentially answer these questions. In Section 2 we prove that (1) holds whenever $0 < p < r$ and n is large. Next, in Section 3, we describe the asymptotic structure of K_{r+1}-free graphs G of order n such that $f(p, G) = \phi(r, p, n)$. We deduce that, if $p \geq r + \lceil \sqrt{2r} \rceil$ and n is large, then

$$\phi(r, p, n) > (1 + \varepsilon) f(p, T_r(n))$$

for some $\varepsilon = \varepsilon(r) > 0$. This disproves Conjecture 6.2 in [3]. In particular,

$$\frac{r}{pe} \geq \frac{\phi(r, p, n)}{n^{p+1}} \geq \frac{r-1}{(p+1)e}$$

holds for large n, and therefore, for any fixed $r \geq 2$,

$$\lim_{n\to\infty} \frac{\phi(r, p, n)}{f(p, T_r(n))}$$

grows exponentially in p.

The case $r = 2$ is considered in detail in Section 4; we show that, if $r = 2$, equality (1) holds for $0 < p \leq 3$, and is false for every $p > 3$ and n large.

In Section 5 we extend the above setup. For a fixed $(r+1)$-chromatic graph H, $(r \geq 2)$, let $\phi(H, p, n)$ be the maximum value of $f(p, G)$ taken over all H-free graphs G of order n. It turns out that, for every r and p,

$$\phi(H, p, n) = \phi(r, p, n) + o\left(n^{p+1}\right). \tag{2}$$

This result completely settles, with the proper changes, Conjecture 6.1 of [3]. In fact, Pikhurko [5] proved this for $p \geq 1$, although he incorrectly assumed that (1) holds for all sufficiently large n.

2 The function $\phi(r, p, n)$ for $p < r$

In this section we shall prove the following theorem.

Theorem 1 *For every $r \geq 2$, $0 < p < r$, and sufficiently large n,*

$$\phi(r, p, n) = f(p, T_r(n)).$$

Proof Erdős [2] proved that, for every K_{r+1}-free graph G, there exists an r-partite graph H with $V(H) = V(G)$ such that $d_G(u) \leq d_H(u)$ for every $u \in V(G)$. As Caro and Yuster noticed, this implies that, for K_{r+1}-free graphs G of order n, if $f(p, G)$ attains a maximum then G is a complete r-partite graph. Every complete r-partite graph is defined uniquely by the size of its vertex

classes, that is, by a vector $(n_i)_1^r$ of positive integers satisfying $n_1 + ... + n_r = n$; note that the Turán graph $T_r(n)$ is uniquely characterized by the condition $|n_i - n_j| \leq 1$ for every $i, j \in [r]$. Thus we have

$$\phi(r,p,n) = \max\left\{\sum_{i=1}^{r} n_i (n-n_i)^p \ : \ n_1 + ... + n_r = n, \ 1 \leq n_1 \leq ... \leq n_r\right\}. \tag{3}$$

Let $(n_i)_1^r$ be a vector on which the value of $\phi(r,p,n)$ is attained. Routine calculations show that the function $x(n-x)^p$ increases for $0 \leq x \leq \frac{n}{p+1}$, decreases for $\frac{n}{p+1} \leq x \leq n$, and is concave for $\frac{2n}{p+1} \leq x \leq n$. If $n_r \leq \left\lfloor \frac{2n}{p+1} \right\rfloor$, the concavity of $x(n-x)^p$ implies that $n_1 - n_r \leq 1$, and the proof is completed, so we shall assume $n_r > \left\lfloor \frac{2n}{p+1} \right\rfloor$. Hence we deduce

$$n_1(r-1) + \left\lfloor \frac{2n}{p+1} \right\rfloor < \ n_1 + ... + n_r = n. \tag{4}$$

We shall also assume

$$n_1 \geq \left\lfloor \frac{n}{p+1} \right\rfloor, \tag{5}$$

since otherwise, adding 1 to n_r and subtracting 1 from n_1, the value $\sum_{i=1}^{r} n_i (n-n_i)^p$ will increase, contradicting the choice of $(n_i)_1^r$. Notice that, as $n_1 \leq n/r$, inequality (5) is enough to prove the assertion for $p \leq r-1$ and every n. From (4) and (5), we obtain that

$$(r-1)\left\lfloor \frac{n}{p+1} \right\rfloor + \left\lfloor \frac{2n}{p+1} \right\rfloor < n.$$

Letting $n \to \infty$, we see that $p \geq r$, contradicting the assumption and completing the proof. □

Maximizing independently each summand in (3), we see that, for every $r \geq 2$ and $p > 0$,

$$\phi(r,p,n) \leq \frac{r}{p+1}\left(\frac{p}{p+1}\right)^p n^{p+1}. \tag{6}$$

3 The asymptotics of $\phi(r,p,n)$

In this section we find the asymptotic structure of K_{r+1}-free graphs G of order n satisfying $f(p,G) = \phi(r,p,n)$, and deduce asymptotic bounds on $\phi(r,p,n)$.

Theorem 2 *For all $r \geq 2$ and $p > 0$, there exists $c = c(p,r)$ such that the following assertion holds.*

If $f(p,G) = \phi(r,p,n)$ for some K_{r+1}-free graph G of order n, then G is a complete r-partite graph having $r-1$ vertex classes of size $cn + o(n)$.

Proof We already know that G is a complete r-partite graph; let $n_1 \leq ... \leq n_r$ be the sizes of its vertex classes and, for every $i \in [r]$, set $y_i = n_i/n$. It is easy to see that

$$\phi(r,p,n) = \psi(r,p)\, n^{p+1} + o\left(n^{p+1}\right),$$

where the function $\psi(r,p)$ is defined as

$$\psi(r,p) = \max\left\{ \sum_{i=1}^{r} x_i (1-x_i)^p \ : \ x_1 + ... + x_r = 1,\ 0 \leq x_1 \leq ... \leq x_r \right\}$$

We shall show that if the above maximum is attained at $(x_i)_1^r$, then $x_1 = ... = x_{r-1}$. Indeed, the function $x(1-x)^p$ is concave for $0 \leq x \leq 2/(p+1)$, and convex for $2/(p+1) \leq x \leq 1$. Hence, there is at most one x_i in the interval $(2/(p+1) \leq x \leq 1]$, which can only be x_r. Thus $x_1, ..., x_{r-1}$ are all in the interval $[0, 2/(p+1)]$, and so, by the concavity of $x(1-x)^p$, they are equal. We conclude that, if

$$0 \leq x_1 \leq ... \leq x_r,\ x_1 + ... + x_r = 1,$$

and $x_j > x_i$ for some $1 \leq i < j \leq r-1$, then $\sum_{i=1}^{r} x_i (1-x_i)^p$ is below its maximum value. Applying this conclusion to the numbers $(y_i)_1^r$, we deduce the assertion of the theorem. $\square$

Set

$$g(r,p,x) = (r-1)\, x (1-x)^p + (1-(r-1)\, x)\,(rx)^p.$$

From the previous theorem it follows that

$$\psi(r,p) = \max_{0 \leq x \leq 1/(r-1)} g(r,p,x).$$

Finding $\psi(r,p)$ is not easy when $p > r$. In fact, for some $p > r$, there exist $0 < x < y < 1$ such that

$$\psi(r,p) = g(r,p,x) = g(r,p,y).$$

In view of the original claim concerning (1), it is somewhat surprising, that for $p > 2r-1$, the point $x = 1/r$, corresponding to the Turán graph, not only fails to be a maximum of $g(r,p,x)$, but, in fact, is a local minimum.

Observe that

$$\frac{f(p, T_r(n))}{n^{p+1}} = \left(\frac{r-1}{r}\right)^p + o(1),$$

so, to find for which p the function $\phi(r,p,n)$ is significantly greater than $f(p, T_r(n))$, we shall compare $\psi(r,p)$ to $\left(\frac{r-1}{r}\right)^p$.

Theorem 3 *Let $r \geq 2$, $p \geq r + \left\lceil \sqrt{2r} \right\rceil$. Then*

$$\psi(r,p) > (1+\varepsilon)\left(\frac{r-1}{r}\right)^p.$$

for some $\varepsilon = \varepsilon(r) > 0$.

Proof We have

$$\psi(r,p) \geq g\left(r,p,\frac{1}{p}\right) = \frac{r-1}{p}\left(\frac{p-1}{p}\right)^p + \left(1-\frac{r-1}{p}\right)\left(\frac{r-1}{p}\right)^p$$
$$> \frac{r-1}{p}\left(\frac{p-1}{p}\right)^p.$$

To prove the theorem, it suffices to show that

$$\frac{r-1}{p}\left(\frac{(p-1)\,r}{p\,(r-1)}\right)^p > 1+\varepsilon \qquad (7)$$

for some $\varepsilon = \varepsilon(r) > 0$. Routine calculations show that

$$\frac{r-1}{p}\left(1+\frac{p-r}{p\,(r-1)}\right)^p$$

increases with p. Thus, setting $q = \lceil\sqrt{2r}\rceil$, we find that

$$\frac{r-1}{p}\left(1+\frac{p-r}{p\,(r-1)}\right)^p$$
$$\geq \frac{r-1}{r+q}\left(1+\binom{r+q}{1}\frac{q}{(r+q)\,(r-1)} + \binom{(r+q)}{2}\frac{q^2}{(r+q)^2\,(r-1)^2}\right)$$
$$= \frac{r-1}{r+q} + \frac{q}{r+q} + \frac{q^2\,(r+q-1)}{2\,(r+q)^2\,(r-1)} \geq 1 - \frac{1}{r+q} + \frac{r\,(r+q-1)}{(r+q)^2\,(r-1)}$$
$$= 1 + \frac{r\,(r+q-1) - (r+q)\,(r-1)}{(r+q)^2\,(r-1)} = 1 + \frac{q}{(r+q)^2\,(r-1)}.$$

Hence, (7) holds with

$$\varepsilon = \frac{\lceil\sqrt{2r}\rceil}{\left(r+\lceil\sqrt{2r}\rceil\right)^2\,(r-1)},$$

completing the proof. $\square$

We have, for n sufficiently large,

$$\frac{\phi(r,p,n)}{n^{p+1}} = \psi(r,p) + o(1) \geq g\left(r,p,\frac{1}{p+1}\right) + o(1)$$
$$= \frac{r-1}{p+1}\left(\frac{p}{p+1}\right)^p + \left(1-\frac{r-1}{p+1}\right)\left(\frac{r-1}{p+1}\right)^p + o(1)$$
$$> \frac{r-1}{p+1}\left(\frac{p}{p+1}\right)^p.$$

Hence, in view of (6), we find that, for n large,

$$\frac{r}{pe} \geq \frac{r}{p}\left(\frac{p}{p+1}\right)^{p+1} \geq \frac{\phi(r,p,n)}{n^{p+1}} \geq \frac{r-1}{p+1}\left(\frac{p}{p+1}\right)^p \geq \frac{(r-1)}{(p+1)\,e}.$$

In particular, we deduce that, for any fixed $r \geq 2$,

$$\lim_{n \to \infty} \frac{\phi(r,p,n)}{f(p,T_r(n))}$$

grows exponentially in p.

4 Triangle-free graphs

For triangle-free graphs, i.e., $r = 2$, we are able to pinpoint the value of p for which (1) fails, as stated in the following theorem.

Theorem 4 *If $0 < p \leq 3$ then*

$$\phi(3,p,n) = f(p,T_2(n)). \tag{8}$$

For every $\varepsilon > 0$, there exists δ such that if $p > 3 + \delta$ then

$$\phi(3,p,n) > (1+\varepsilon) f(p,T_2(n)) \tag{9}$$

for n sufficiently large.

Proof We start by proving (8). From the proof of Theorem 1 we know that

$$\phi(p,n,3) = \max_{k \in \lceil n/2 \rceil} \{k(n-k)^p + (n-k)k^p\}.$$

Our goal is to prove that the above maximum is attained at $k = \lceil n/2 \rceil$.

If $0 < p \leq 2$, the function $x(1-x)^p$ is concave, and (8) follows immediately.

Next, assume that $2 < p \leq 3$; we claim that the function

$$g(x) = (1+x)(1-x)^p + (1-x)(1+x)^p$$

is concave for $|x| \leq 1$. Indeed, we have

$$\begin{aligned} g(x) &= \left(1-x^2\right)\left((1-x)^{p-1} + (1+x)^{p-1}\right) = 2\left(1-x^2\right)\sum_{n=0}^{\infty}\binom{p-1}{2n}x^{2n} \\ &= 2 + 2\sum_{n=1}^{\infty}\left(\binom{p-1}{2n} - \binom{p-1}{2n-2}\right)x^{2n} \\ &= 2 + 2\sum_{n=1}^{\infty}\binom{p-1}{2n-2}\left(\frac{(p-2n-1)(p-2n-2)}{(2n-1)2n} - 1\right)x^{2n}. \end{aligned}$$

Since, for every n, the coefficient of x^{2n} is nonpositive, the function $g(x)$ is concave, as claimed.

Therefore, the function $h(x) = x(n-x)^p + (n-x)x^p$ is concave for $1 \leq x \leq n$. Hence, for every integer $k \in [n]$, we have

$$\begin{aligned} h\left(\left\lceil \frac{n}{2} \right\rceil\right) + h\left(\left\lfloor \frac{n}{2} \right\rfloor\right) &\geq h(k) + h(n-k) = 2h(k) \\ &= 2\left(k(n-k)^p + (n-k)k^p\right), \end{aligned}$$

proving (8).

Inequality (9) follows easily, since, in fact, for every $p > 3$, the function $g(x)$ has a local minimum at 0. $\square$

5 H-free graphs

In this section we are going to prove the following theorem.

Theorem 5 *For every $r \geq 2$, and $p > 0$,*

$$\phi(H, p, n) = \phi(r, p, n) + o\left(n^{p+1}\right).$$

A few words about this theorem seem in place. As already noted, Pikhurko [5] proved the assertion for $p \geq 1$; although he incorrectly assumed that (1) holds for all p and sufficiently large n, his proof is valid, since it is independent of the exact value of $\phi(r, p, n)$. Our proof is close to Pikhurko's, and is given only for the sake of completeness.

We shall need the following theorem (for a proof see, e.g., [1], Theorem 33, p. 132).

Theorem 6 *Suppose H is an $(r+1)$-chromatic graph. Every H-free graph G of sufficiently large order n can be made K_{r+1}-free by removing $o\left(n^2\right)$ edges.*

Proof of Theorem 5 Select a K_{r+1}-free graph G of order n such that $f(p, G) = \phi(r, p, n)$. Since G is r-partite, it is H-free, so we have $\phi(H, p, n) \geq \phi(r, p, n)$. Let now G be a H-free graph of order n such that

$$f(p, G) = \phi(H, p, n).$$

Theorem 6 implies that there exists a K_{r+1}-free graph F that may be obtained from G by removing at most $o\left(n^2\right)$ edges. Obviously, we have

$$e(G) = e(F) + o\left(n^2\right) \leq \frac{r-1}{2r} n^2 + o\left(n^2\right).$$

For $0 < p \leq 1$, by Jensen's inequality, we have

$$\left(\frac{1}{n} f(p, G)\right)^{1/p} \leq \frac{1}{n} f(1, G) = \frac{1}{n} 2e(G) \leq \frac{r-1}{r} n + o(n).$$

Hence, we find that

$$f(p, G) \leq \left(\frac{r-1}{r}\right)^p n^{p+1} + o\left(n^{p+1}\right) = \phi(r, p, n) + o\left(n^{p+1}\right),$$

completing the proof.

Next, assume that $p > 1$. Since the function $xn^{p-1} - x^p$ is decreasing for $0 \leq x \leq n$, we find that

$$d_G^p(u) - d_F^p(u) \leq (d_G(u) - d_F(u))\, n^{p-1}$$

for every $u \in V(G)$. Summing this inequality for all $u \in V(G)$, we obtain

$$\begin{aligned} f(p, G) &\leq f(p, F) + (d_G(u) - d_F(u))\, n^{p-1} = f(p, F) + o\left(n^{p+1}\right) \\ &\leq \phi(r, p, n) + o\left(n^{p+1}\right), \end{aligned}$$

completing the proof. $\square$

6 Concluding remarks

It seems interesting to find, for each $r \geq 3$, the minimum p for which the equality (1) is essentially false for n large. Computer calculations show that this value is roughly 4.9 for $r = 3$, and 6.2 for $r = 4$, suggesting that the answer might not be easy.

References

[1] B. Bollobás, *Modern Graph Theory,* Graduate Texts in Mathematics, **184,** Springer-Verlag, New York (1998), xiv+394 pp.

[2] P. Erdős, On the graph theorem of Turán. (in Hungarian), *Mat. Lapok* **21** (1970), 249–251.

[3] Y. Caro and R. Yuster, A Turán type problem concerning the powers of the degrees of a graph, *Electron. J. Comb.* **7** (2000), RP 47.

[4] R. H. Schelp, review in *Math. Reviews,* MR1785143 (2001f:05085), 2001.

[5] O. Pikhurko, Remarks on a Paper of Y. Caro and R. Yuster on Turán problem, preprint, *arXiv:math.CO/0101235v1* 29 Jan 2001.